<center>**CHAPTER 2:**</center>

<center>*VECTORS IN 3D*</center>

2.1 DEFINITION AND REPRESENTATION OF VECTORS

A vector in three dimensions is a quantity that is determined by its magnitude and direction. Vectors are added and multiplied by numbers in specific ways that are discussed later on in the section.

Vectors in three dimensions can be represented by an arrow. An arrow has components of magnitude and direction: the direction in which it points and the length of the arrow. Hence an arrow is a useful means to represent phenomena that is characterized by its direction and magnitude. For example, if we wish to represent the velocity of an automobile in a given instant of time, the direction of the arrow is the direction in which the automobile is moving at that moment and the length of the vector represents the speed of the automobile.

However the notion of a vector consists <u>only</u> of magnitude and direction. Hence we have to modify our idea of an arrow to think <u>only</u> of its length and direction. This will mean that different arrows of the same length and pointing in the same direction, as in Figure 2.1.1, will represent <u>the same</u> vector. Mathematicians can get around this difficulty by saying that a vector is a set containing all arrows that point in a given
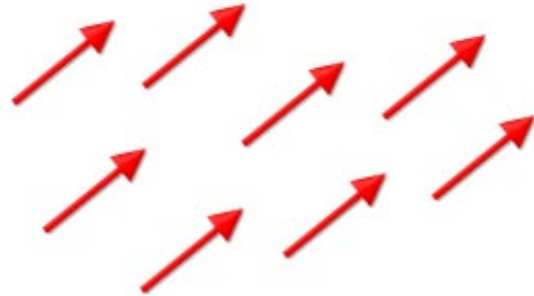

Figura 2.1.1

direction and have a given magnitude. However we will keep it simple and think of a vector as an arrow that can be moved around as long as we don't change its magnitude or direction.

DISPLACEMENT VECTORS

Suppose that a Saint Bernard starts the day at a point *P* and ends the day at a point *Q* that is 10 miles northeast of *P*. Suppose that another Saint Bernard starts the day at a very distant point *R* and ends the day at a point *S* that is 10 miles northeast of *R* (see Figure 2.1.2). If we are only interested in overall displacement, all we care about is that each of the dogs ended the day at a distance of 10 miles from the starting point and in the northeast direction from the starting point. The *displacement vector* of each of the dogs is exactly the same. We don't care how they got from starting point to ending point. Their actual travel paths could be very different. If we only consider starting point and ending point, both dogs moved the same distance in the same direction and hence their displacement vectors are the same.
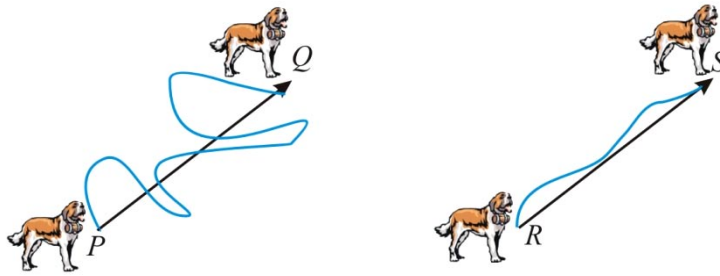
Figure 2.1.2

Having determined what a vector is, our next step is to represent it. If we place an arrow in three dimensions then the magnitude and direction of the arrow will be determined if we are told how many units we must move in the *x* direction, in the *y* direction, and in the *z* direction, in order to get from the tail to the head of the arrow (see Figure 2.1.3). Hence, in three dimensions, the following notation can be used to represent a vector, $\vec{v}$:

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- *a* is the number of units moved in the *x* direction.
- *b* is the number of units moved in the *y* direction.
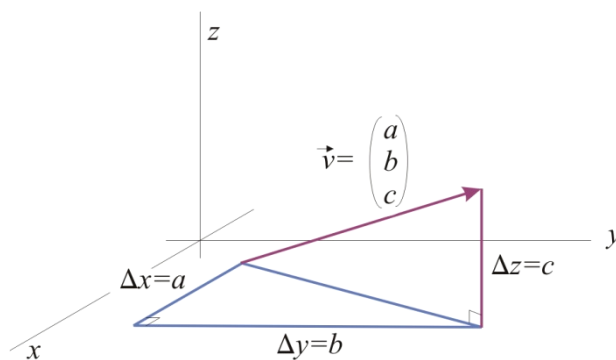- *c* is the number of units moved in the *z* direction.
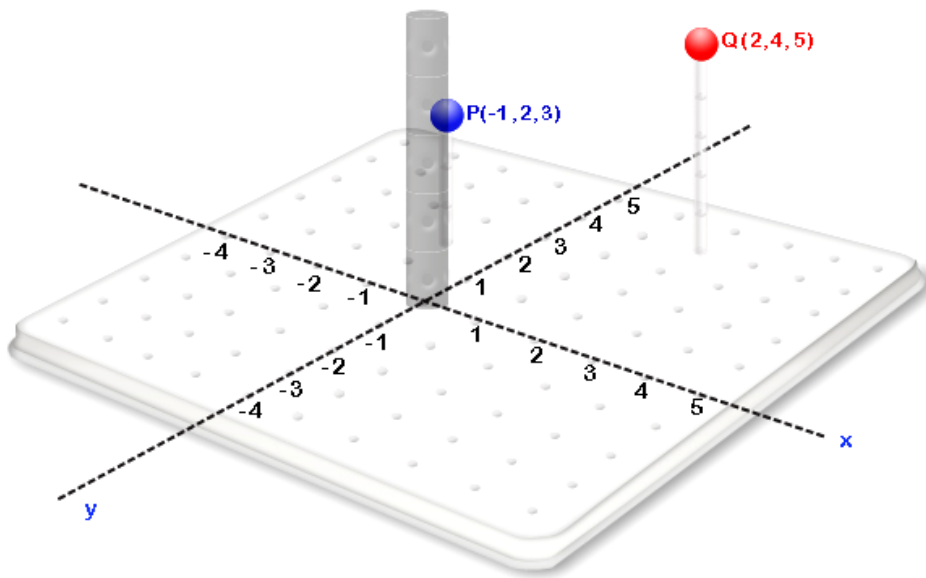

Figure 2.1.3

For convenience, in this text we will use the notation $\vec{v}\langle a, b, c \rangle$.
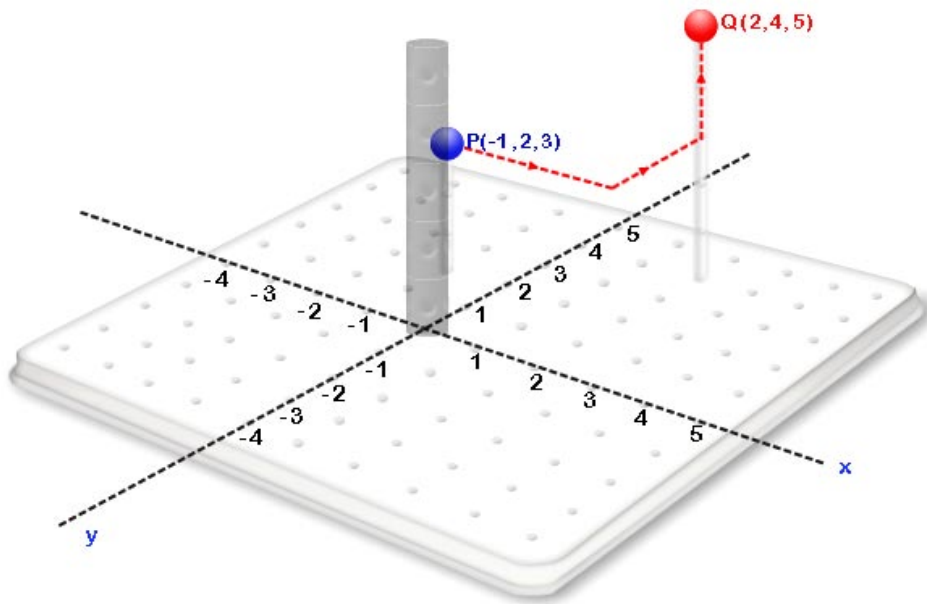
PP 2.1.1

a. Place the points P (-1, 2, 3) and Q (2, 4, 5) in your 3D Kit.

b. Connect the two points with a vector.

c. Obtain the displacement in x, the displacement in y, and the displacement z for the vector and use this to find the vector $< a, b, c >$ that represents the displacement from P (-1, 2, 3) to Q (2, 4, 5).

Solution

a.

b.

c. $< 3, 2, 2 >$

PP 2.1.2

Reflect on the previous practice problem and examples and think of a formula to obtain the vector representing the displacement from $P (x_1, y_1, z_1)$ to $Q (x_2, y_2, z_2)$.

Solution

$< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$

PP 2.1.3

a. With your 3D Kit, place the start (tail) of the vector $< 1, 3, 3 >$ at the point $(1, 1, 2)$. What are the coordinates of the point where the vector ends?

b. Now continue by placing the start vector $< 2, -4, 1 >$ where the vector $< 1, 3, 3 >$ ends. What are the coordinates of the point where the vector $< 2, -4, 1 >$ ends?

c. Place a vector from the point $< 1, 1, 2 >$, where $< 1, 3, 3 >$ commences, to the point obtained in (b), where $< 2, -4, 1 >$ ends. This is the vector $< 1, 3, 3 > + < 2, -4, 1 >$. Find the representation $< a, b, c >$ for this vector.
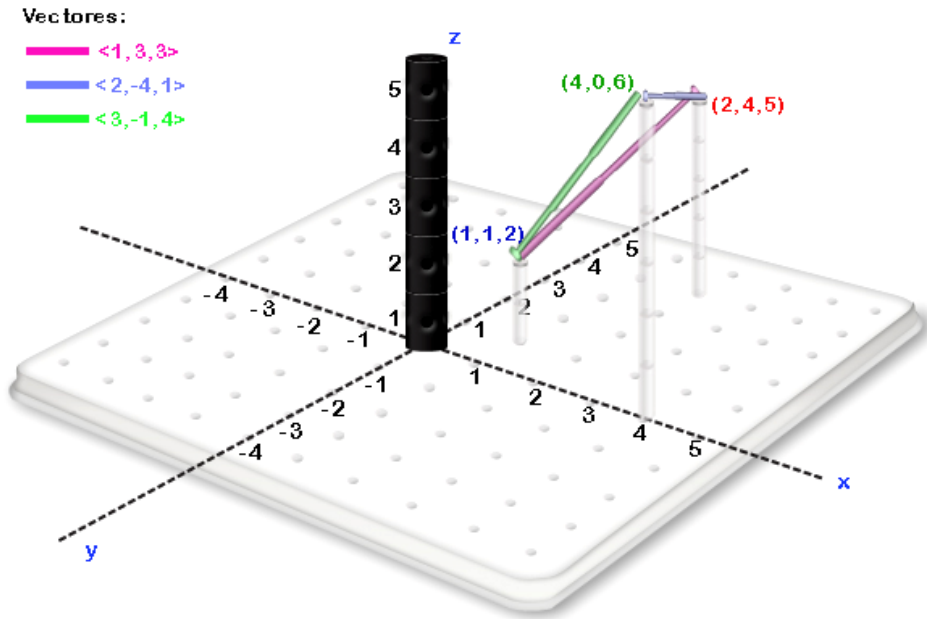
Solution
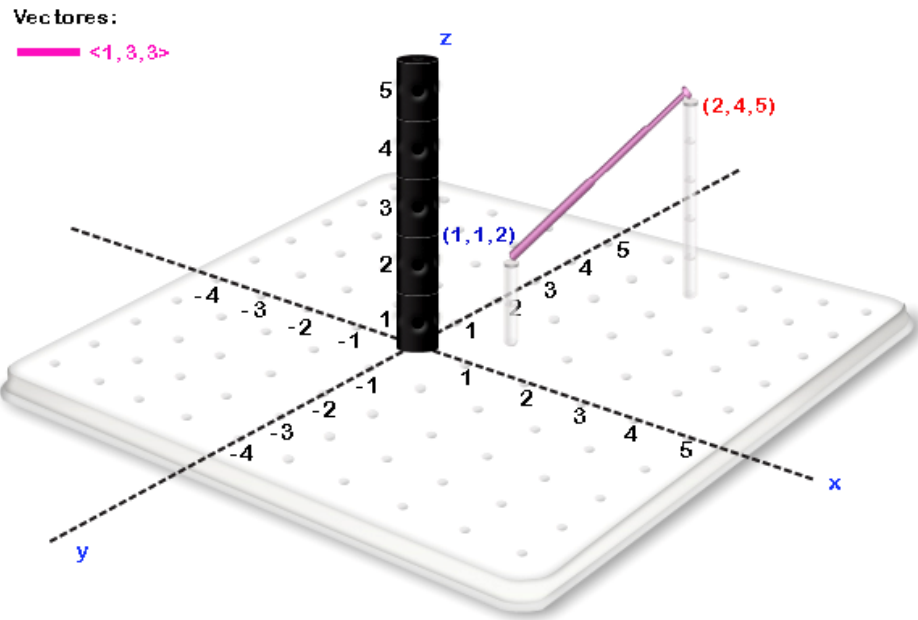
a.
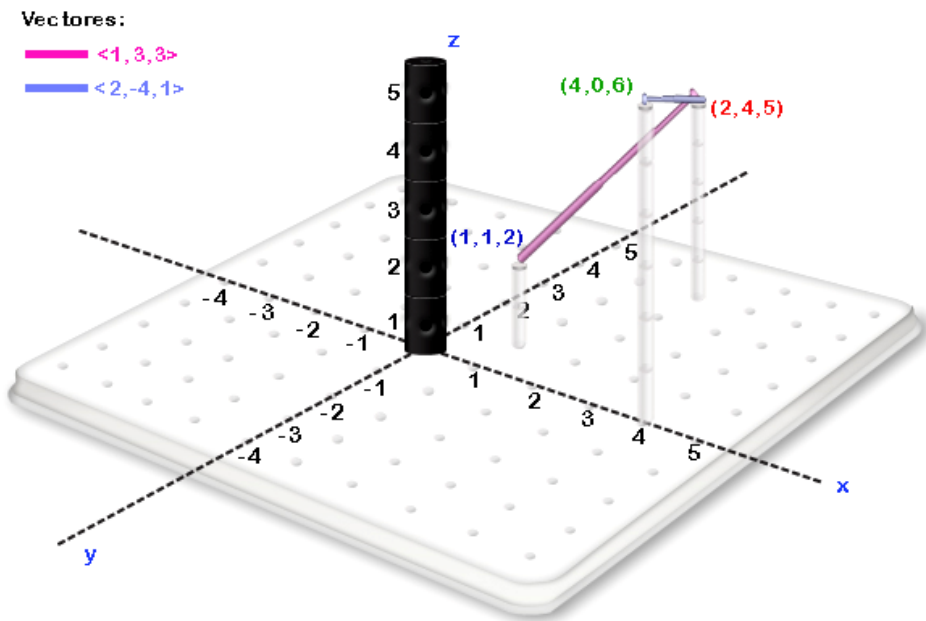
b.

c.

2.2 OPERATIONS WITH VECTORS

THE SUM OF TWO VECTORS

Suppose a person starts at a point P and moves 3 miles to the north. This displacement can be represented with a vector $\vec{a}$. Now suppose that the person moves 4 miles to the east to arrive at a point *R*. This second displacement is represented with a vector $\vec{b}$. The total displacement from point *P* to point *R* is the vector that we call $\vec{a}+\vec{b}$. As Figure 2.2.1 shows, the magnitude of $\vec{a}+\vec{b}$ is 5 miles (<u>not</u> 7 miles) and the direction can be described as $\tan(4/3) \approx 53$ east of due north.

Vector addition is defined so that if $\vec{a}$ and $\vec{b}$ are displacement vectors then $\vec{a}+\vec{b}$ is the vector that represents a displacement of $\vec{a}$ followed by a displacement of $\vec{b}$. Given vectors $\vec{a}$ and $\vec{b}$ their sum, $\vec{a}+\vec{b}$, is formed as follows: represent the vector $\vec{a}$ in three-dimensional space, so that it starts at some point *P* and ends at some point *Q*. Now take vector $\vec{b}$ and place it so that it starts where vector $\vec{a}$ ends, that is, so it will start at *Q* and end at some point *R*. The resulting vector that passes directly from *P* to *R* is denoted $\vec{a}+\vec{b}$. Figures 2.2.1 and 2.2.2 both show vector additions.
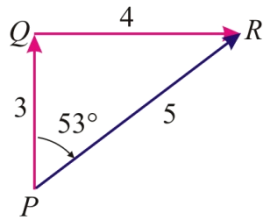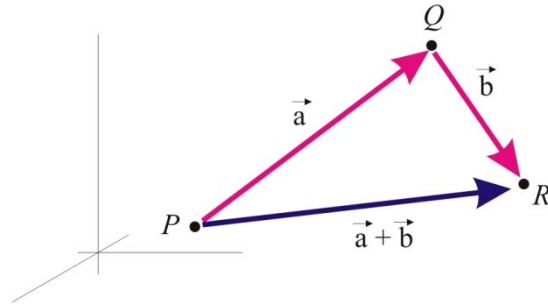
Figure 2.2.1

Figure 2.2.2

Example Exercise 2.2.1: To Obtain the Algebra from the Geometry

If we have the vectors $\vec{a} = \langle 2,3,2 \rangle$ and $\langle 1,2,1 \rangle$, they can be placed end to end. Vector $\vec{a}$ has a displacement of two units to the east and $\vec{b}$ a displacement of one unit to the east. Correspondingly, the net displacement in *x* of $\vec{a}+\vec{b}$ is three units to the east. Vector $\vec{a}$ has a displacement of three units north and $\vec{b}$ of two units south. Correspondingly, the net displacement in *y* of $\vec{a}+\vec{b}$ is 1 unit north. Vector $\vec{a}$ has a displacement of two units up and $\vec{b}$ has a displacement of one unit down. Correspondingly, the net displacement in *z* of $\vec{a}+\vec{b}$ is 1 unit up. Putting it all together we can conclude that $\vec{a}+\vec{b}$ = $\langle 2,3,2 \rangle + \langle 1,2,1 \rangle = \langle 3,1,1 \rangle$.

MULTIPLYING A VECTOR BY A POSITIVE SCALAR

If four copies of a vector $\vec{a}$ are placed end to end, the resulting sum is denoted $\vec{a}+\vec{a}+\vec{a}+\vec{a}$ = $4\vec{a}$. The net effect of this operation is a vector with the same direction as $\vec{a}$ and a length four times greater than the length of $\vec{a}$ (see Figure 2.2.3). Also, if we take a vector with the same direction as $\vec{a}$ and a length half as long as the length of $\vec{a}$, the resulting vector is denoted $(1/2)\vec{a}$ (see Figure 2.2.4). In general, if a vector is multiplied by a positive number *k*, the resulting vector will have the same direction with length altered by a factor of *k*. Borrowing terminology from physics, a number *k* is called a *scalar*.
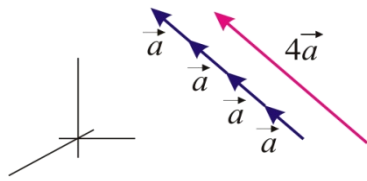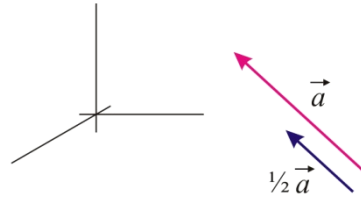
Figure 2.2.3   *Figure 2.2.4*

Example Exercise 2.2.2: To Obtain the Algebra from the Geometry

If we have the vector $\vec{a} = \langle 1,1,1 \rangle$, three vectors $\vec{a}s$ can be placed end to end as shown in Figure 2.2.5 to obtain $3\vec{a}$. One $\vec{a}$ has a movement of one unit to the east so three $\vec{a}s$ in succession will have a movement of three units to the east. Correspondingly the net movement in *x* of $3\vec{a}$ is three units to the east. One $\vec{a}$ has a movement of one unit to the north so three $\vec{a}s$ in succession will have a movement of three units north. Correspondingly the net movement in *y* of $3\vec{a}$ is three units north. One $\vec{a}$ has a movement of one unit downward so three $\vec{a}s$ in succession will have a movement of three units downward. Correspondingly the net movement in *z* of $3\vec{a}$ is three units downward. Hence $3\vec{a} = \langle 1,1,1 \rangle = \langle 3,3,3 \rangle$.
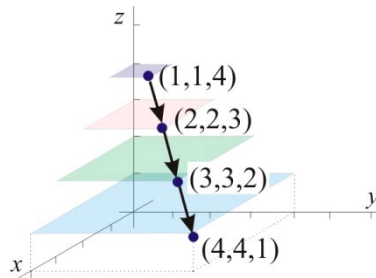

Figure 2.2.5

Generalization

In general we have:

For example:

$3\langle 1,2,3\rangle = \langle 3,6,9\rangle$

$(1/2)\langle 4,8,2\rangle = \langle 2,4,1\rangle$

MULTIPLYING A VECTOR BY -1

If we start with an arrow representing vector $\vec{a}$ and do nothing more than change the direction of the arrow, the resulting vector is denoted $-\vec{a}$ or $-1\,\vec{a}$ (see Figure 2.2.6).
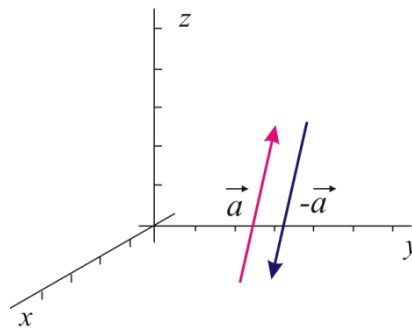

Figure 2.2.6

Example Exercise 2.2.3: To Obtain the Algebra from the Geometry

Assume that we have placed the vector $\vec{a} = \langle 1,2,3\rangle$ so that it goes from $(1,1,0)$ to $(2,3,3)$ as in Figure 2.2.7. If we simply change the direction of $\vec{a}$, then, as shown in Figure 2.2.8, the resulting vector $-\vec{a}$ can be placed to go from $(2,3,3)$ to $(1,1,0)$. The displacement in the x direction of $\vec{a}$ is one unit to the east. If we change the direction, the displacement in the x

direction of $-\vec{a}$ is one unit to the west. The displacement in the y direction of $\vec{a}$ is two units north. If we change the direction, the displacement in the y direction of $-\vec{a}$ is two units south. The displacement in the z direction of $\vec{a}$ is three units upward. If we change the direction, the displacement in the z direction of $-\vec{a}$ is three units downward. Putting all of this together, we can conclude that $-\vec{a} = \langle -1, -2, -3 \rangle$.
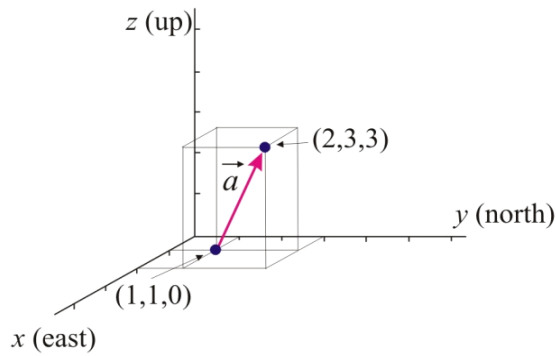

Figure 2.2.7
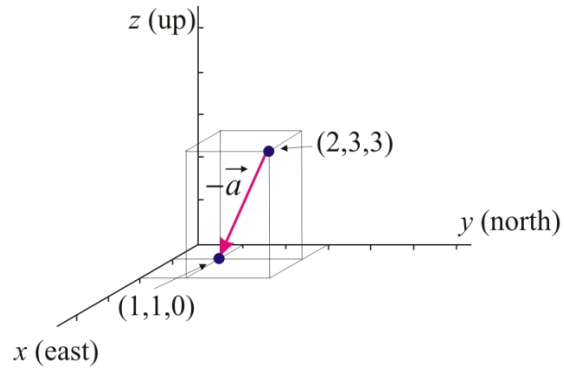
Figure 2.2.8

To summarize:

Now we can define vector subtraction:

Example Exercise 2.2.4: Given vectors $\vec{a}$ and $\vec{b}$ in Figure 2.2.9, represent $\vec{a}-\vec{b}$ geometrically.

Observe that $\vec{a}-\vec{b}=-\vec{b}+\vec{a}$ and think of this last vector in terms of displacement vectors. First do the displacement $-\vec{b}$ and then follow it with $\vec{a}$. So you'd start at the head of $\vec{b}$, travel in the direction opposite to $\vec{b}$ and then travel in the direction of $\vec{a}$ (see Figure 2.2.10).
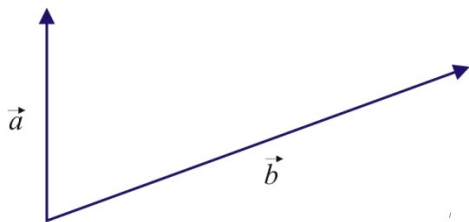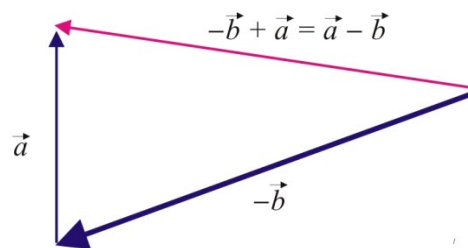

Figure 2.2.9

Figure 2.2.10

Example

$$\langle 4,2,3 \rangle - \langle 1,1,1 \rangle = \langle 2,1,1 \rangle \quad \langle \qquad \rangle$$

$$\langle -3,5,22 \rangle - \langle 2,1,5 \rangle = \langle 3,3 \rangle \quad \langle \qquad \rangle$$

MULTIPLICATION BY A NEGATIVE CONSTANT

Suppose $k>0$. To multiply a vector $\vec{a}$ by the negative constant $-k$, we can divide the process into two parts.

- **Step 1:** $k\vec{a}$ stretches or shrinks the vector $\vec{a}$ to k times its initial length.
- **Step 2:** $-k\vec{a}=-1\cdot k\vec{a}$ changes the direction of the vector $k\vec{a}$.

Geometrically, this is illustrated in Figure 2.2.11 and Figure 2.2.12 where we first stretch or shrink (if $k>1$ we stretch, if $0<k<1$ we shrink) $\vec{a}$ to k times its initial length and then we change its direction.
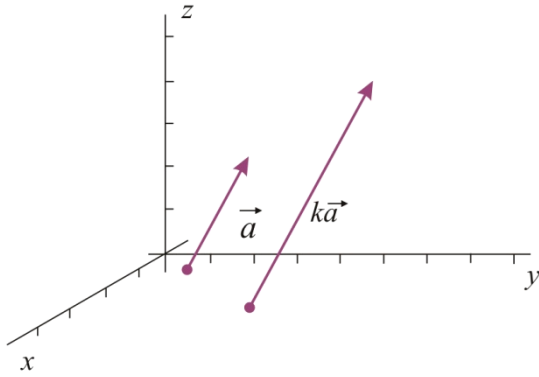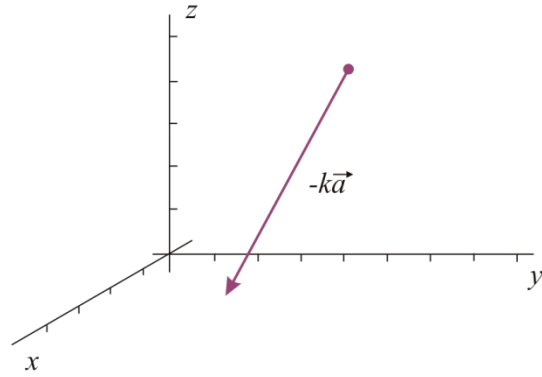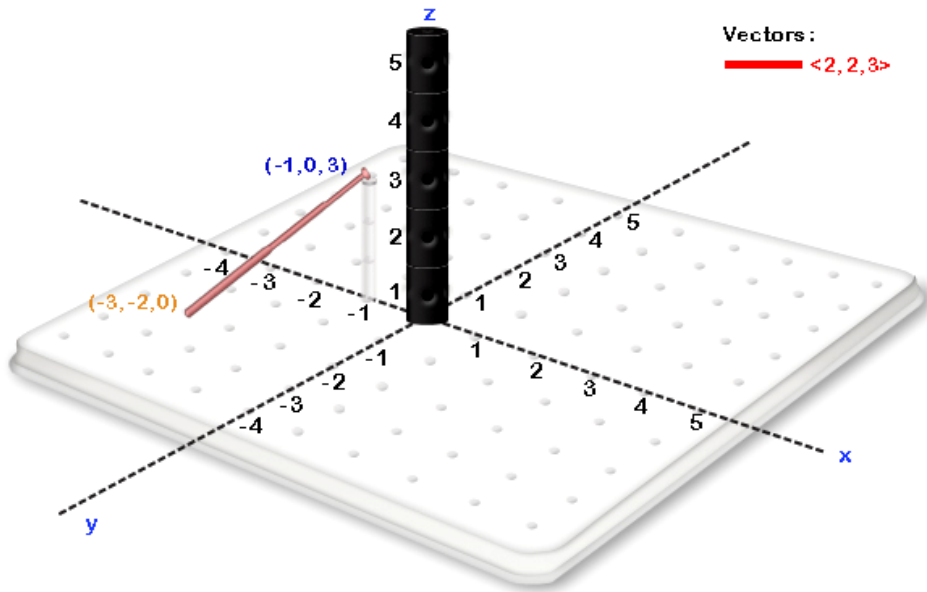
Figure 2.2.11
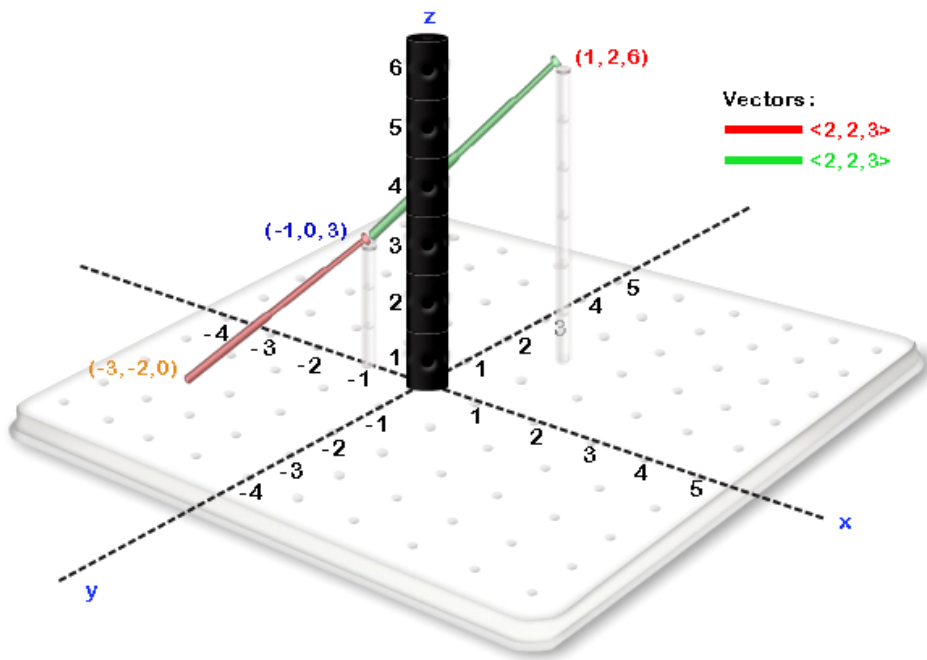

Figure 2.2.12

Practice Problems

PP 2.2.1

a. With your 3D Kit, place the start of vector $< 2, 2, 3 >$ at the point (-3, -2, 0). What are the coordinates of the point where the vector ends?

b. Place a second copy of vector $< 2, 2, 3 >$ where the vector $< 2, 2, 3 >$ ends. What are the coordinates of the point where the vector $< 2, 2, 3 >$ ends?

c. Place a third copy of vector $< 2, 2, 3 >$, starting where the second vector $< 2, 2, 3 >$ ends. What are the coordinates of the point where the third vector $< 2, 2, 3 >$ ends?

d. Place a vector from the point (-3, -2, 0), where the first vector $< 2, 2, 3 >$ commences, to the point obtained in (c) where the third vector $< 2, 2, 3 >$ ends. This is the vector $3 < 2, 2, 3 >$. Find a representation $< a, b, c >$ for the vector $3 < 2, 2, 3 >$

Solution

a.

b.
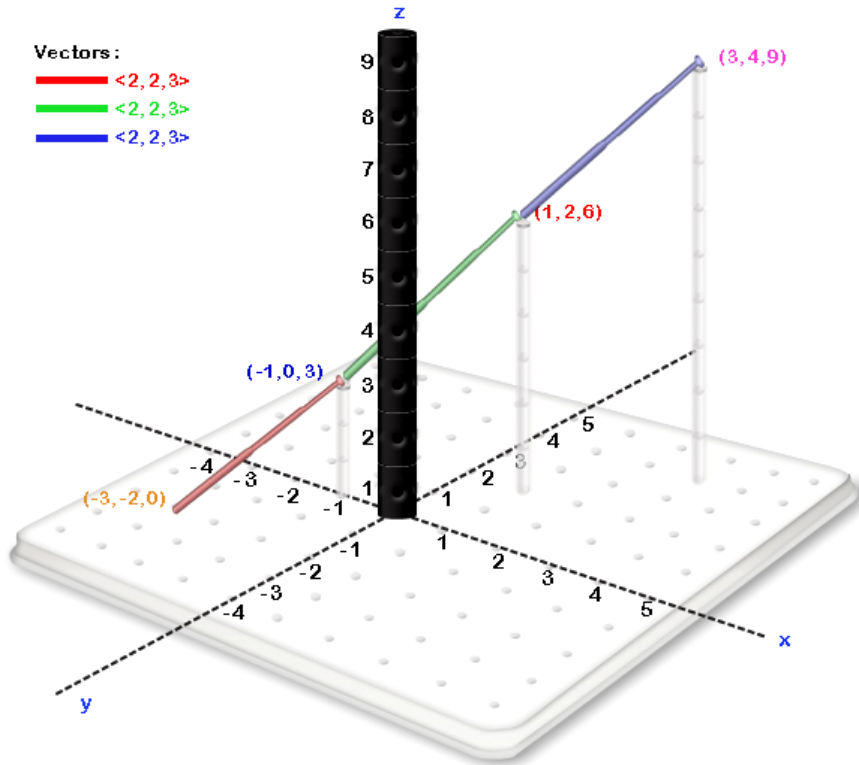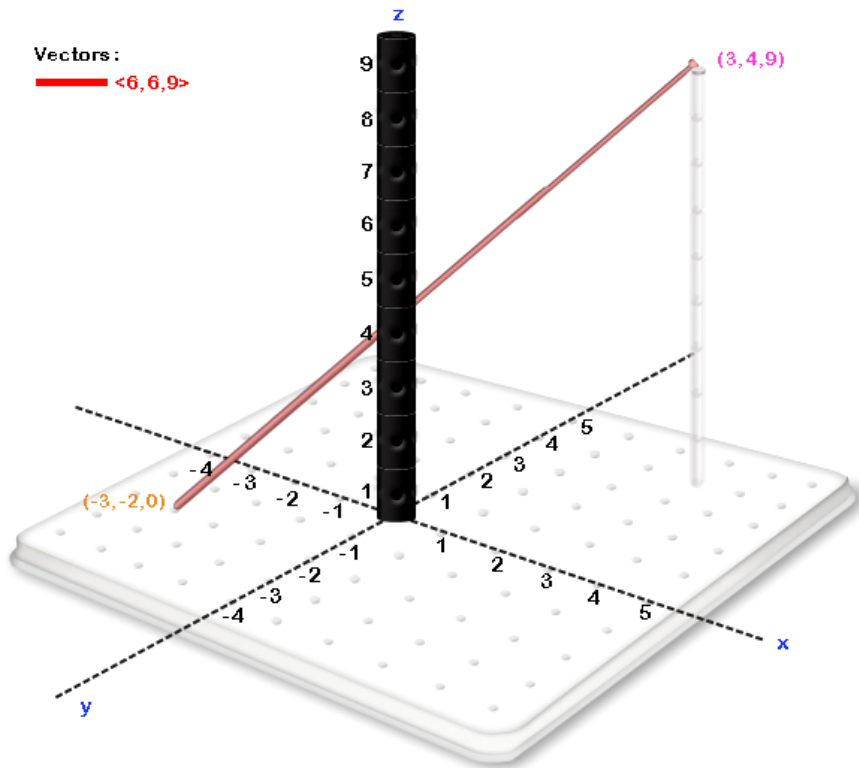
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Vectors:
<2,2,3>
<2,2,3>
<2,2,3>

(3,4,9)
(1,2,6)
(-1,0,3)
(-3,-2,0)

c.

Vectors:
<6,6,9>

(3,4,9)
(-3,-2,0)

d.

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a. Use your 3D Kit to place both vectors $\vec{a} = <3,3,-1>$ and $\vec{b} = <2,-5,3>$ starting at the point P (1, 1, 3).

b. Now starting at the point Q (3, -4, 6) represent a displacement of $-\vec{b}$ followed by a displacement of $\vec{a}$. Where do you end?

c. Use part (b) to represent vector $-\vec{b} + \vec{a}$. Is this different than vector $\vec{a} - \vec{b}$?
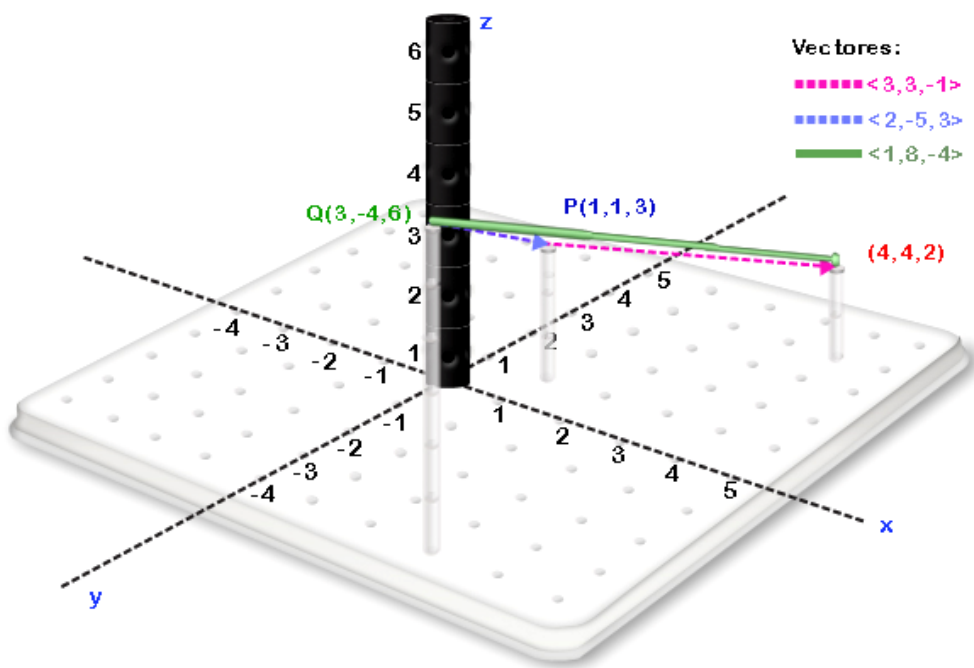
Solution
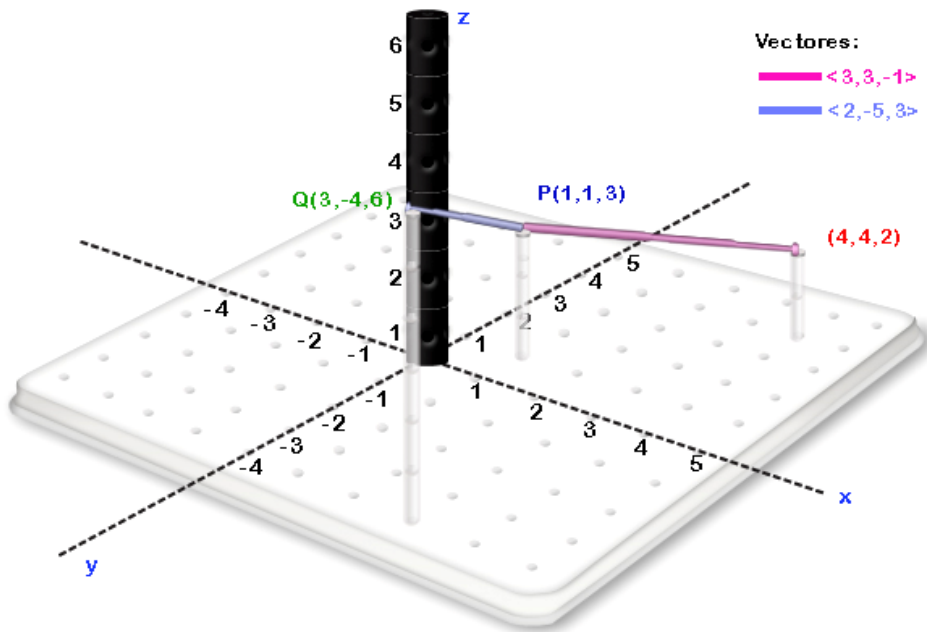
a.
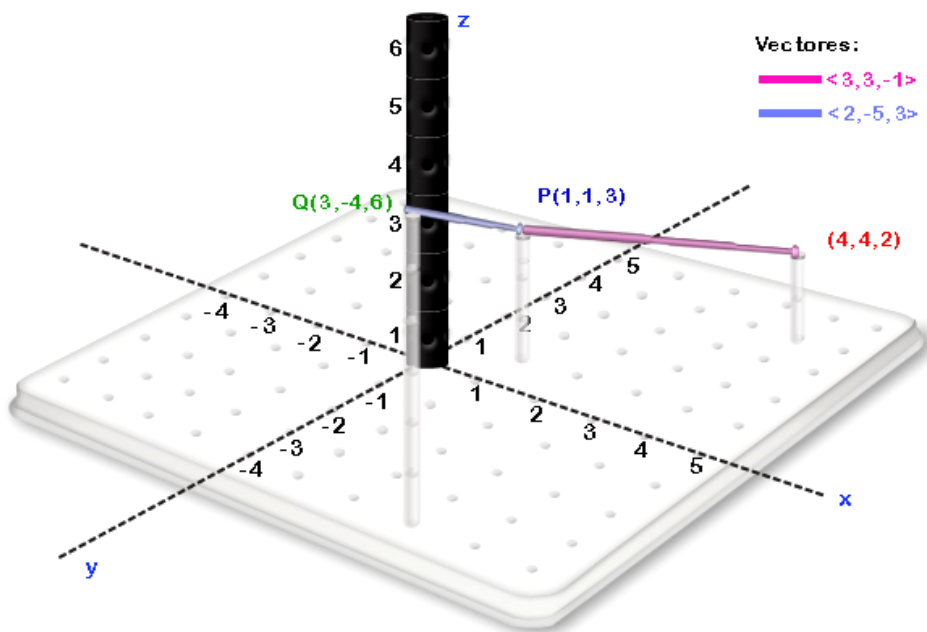
b.


c. Both are the same.

## PP 2.2.3

a. With your 3D Kit, place the start of vector $< 2, 3, -2 >$ at the point (-3, -3, 8). What are the coordinates of the point where the vector ends?

b. Place a second copy of vector $< 2, 3, -2 >$ starting where the first copy of vector $< 2, 3, -2 >$ ended. What are the coordinates of the point where the second vector $< 2, 3, -2 >$ ends?

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c. Place a vector from the point (-3, -3, 8), where the first vector < 2, 3, -2 > commences, to the point obtained in (b), where the second < 2, 3, -2 > ends. This is the vector 2 < 2, 3, -2 >. Find a representation < a, b, c > for the vector 2 < 2, 3, -2 >.

d. Change the direction of the vector obtained in (c). This is the vector -2 < 2, 3, -2 >. Find a representation < a, b, c > for the vector -2 < 2, 3, -2 >.

Solution

a.

b.

c.

Vectors:
<-4,-6,4>

d.
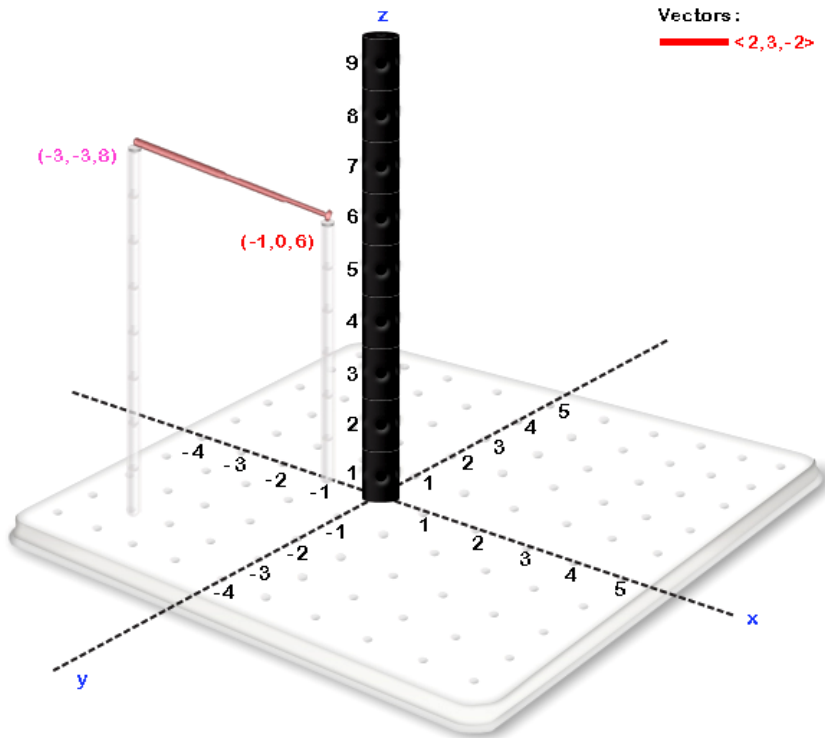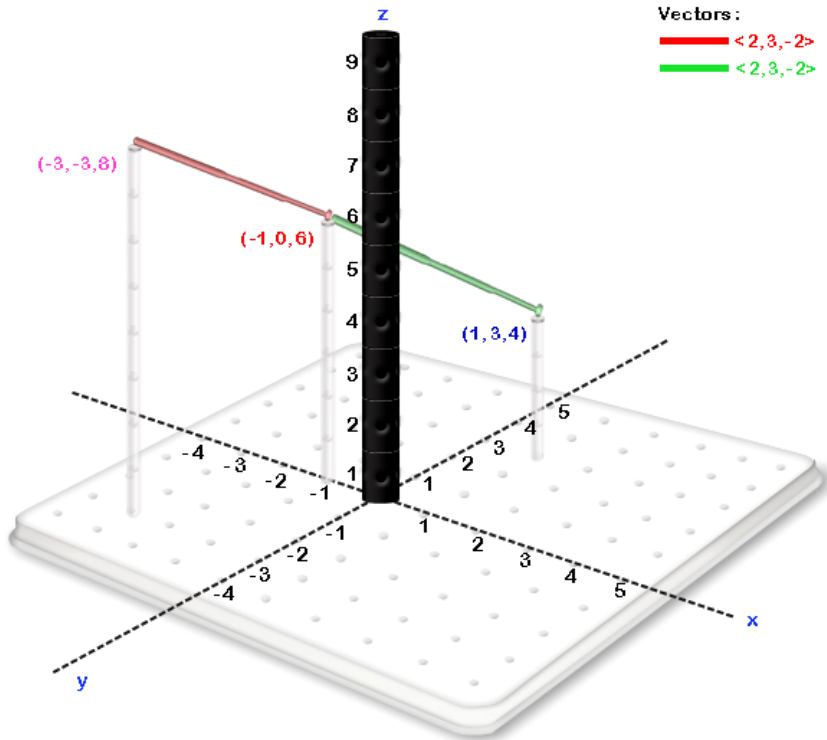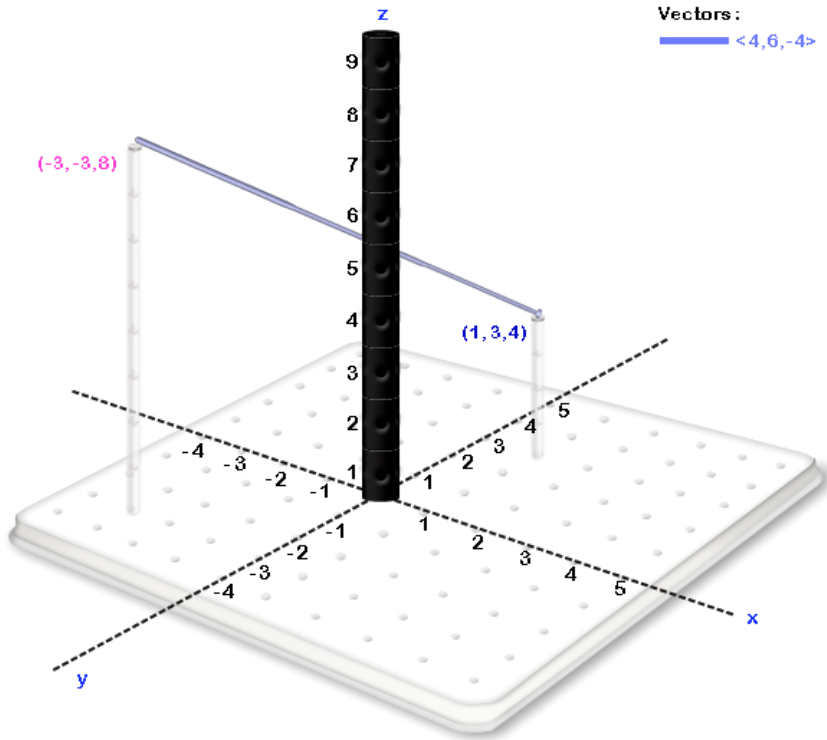
2.3 THE MAGNITUD OF A VECTOR

FINDING THE MAGNITUD OF A VECTOR

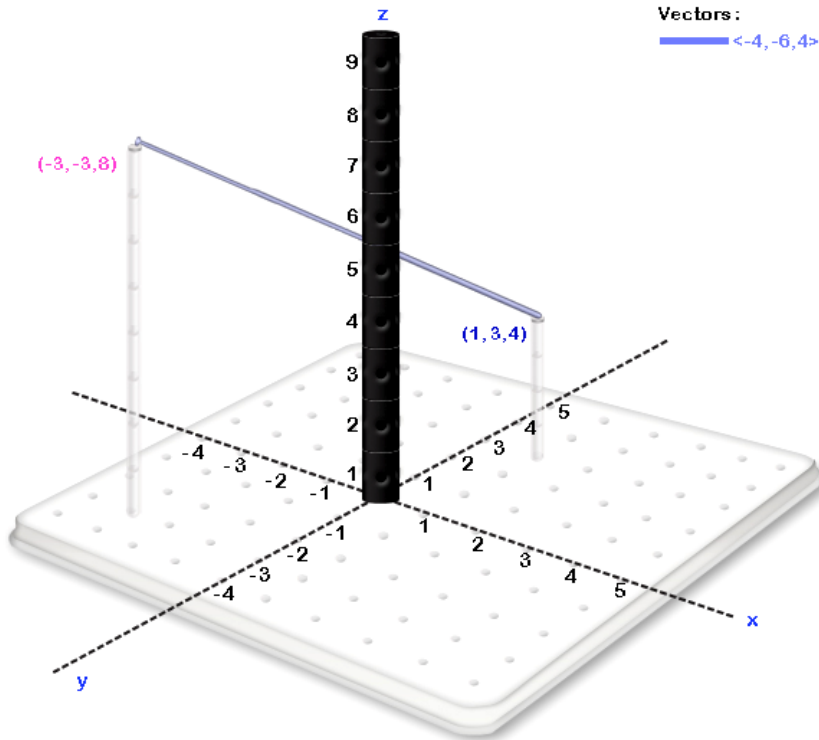The *magnitude* or *length* of a vector $\langle a,b,c \rangle$ is expressed as $\| \langle a,b,c \rangle \|$. The procedure to find this length is fundamentally the same as the procedure to find the distance between two points.

Example Exercise 2.3.1: Find the length of the vector $\langle 3,4,6 \rangle$.

Step 1: As done in Figure 2.3.1, express vector $\langle 3,4,6 \rangle$ as a sum of a vertical vector $\langle 0,0,6 \rangle$ and a horizontal vector $\langle 3,4,0 \rangle$. The vector $\langle 0,0,6 \rangle$ has a length of 6.

Step 2: As shown in Figure 2.3.2, find the length of vector $\langle 3,4,0 \rangle$ in the *xy* plane. As this is a problem in two dimensions, we can use Pythagoras as usual to show that the length of the vector $\langle 3,4,0 \rangle$ in the *xy* plane is equal to 5.

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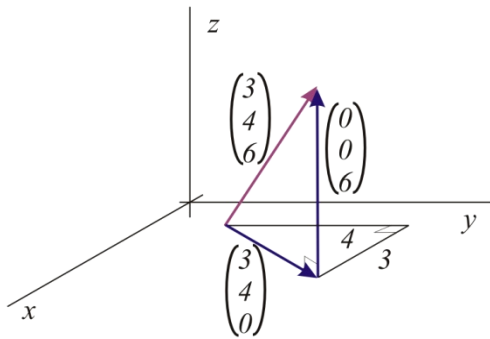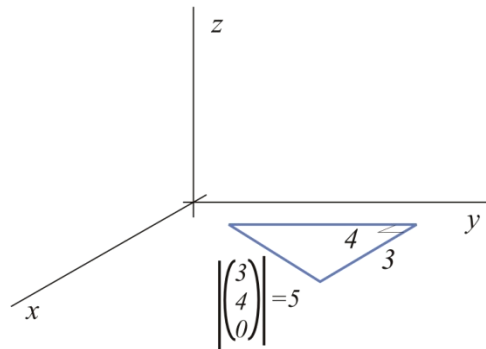
Figure 2.3.1

Figure 2.3.2

Step 3: Use the right triangle in Figure 2.3.3 and the Theorem of Pythagoras to obtain the length $\left\|\langle 3,4,6\rangle\right\|=\sqrt{61}=\sqrt{}$.
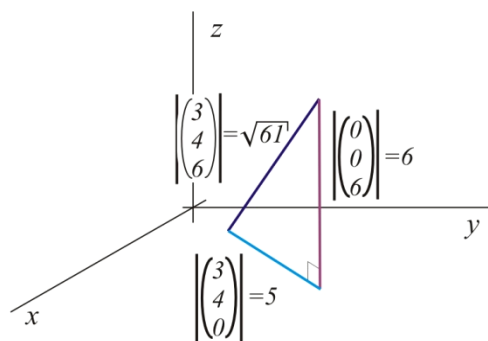
Figure 2.3.3

In general, to find the length of the vector $\langle a,b,c\rangle$:

Step 1: As in Figure 2.3.4, express vector $\langle a,b,c\rangle$ as the sum of vertical vector $\langle 0,0,c\rangle$ and horizontal vector $\langle a,b,0\rangle$. Note that the length of vector $\langle 0,0,c\rangle$ is $|c|$.

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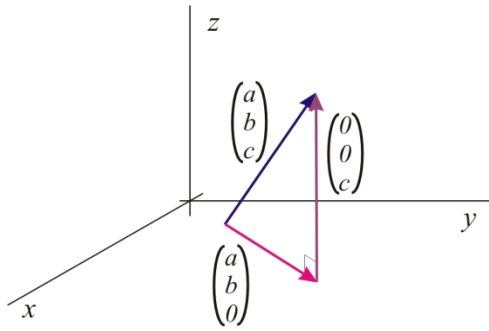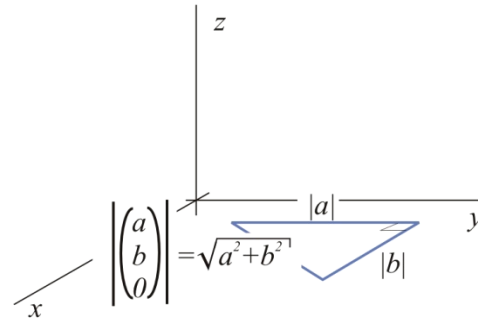
Figure 2.2.4

Figure 2.3.5

Step 2: Find the length of the vector $\langle a, b, 0 \rangle$ in the *xy* plane. As this is a problem in two dimensions, we can use the Theorem of Pythagoras to show that the length of the vector $\langle a, b, 0 \rangle$ in the *xy* plane is equal to $\sqrt{a^2+b^2}$ (see Figure 2.3.5).

Step 3: Finally, we can use Pythagoras on the right triangle in Figure 2.3.6 to quickly conclude that the length of $\langle a, b, c \rangle$ is $\sqrt{\left(\sqrt{a^2+b^2}\right)^2 + |c|^2}$ $\sqrt{}$.
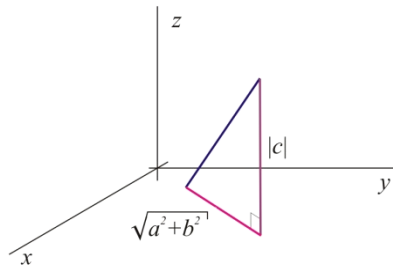
Figure 2.3.6

Hence we have:

Magnitude of a Vector

The magnitude of a vector $\langle a, b, c \rangle$ is $\|\langle a, b, c \rangle\| = \sqrt{a^2+b^2+c^2}$.

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PP 2.3.1

The following problems begin at the point (-1, 0, 0).

a. On the base of your 3D Kit, use Pythagoras to find the length of the vector $< 2, 3, 0 >$.

b. At the end of the vector $< 2, 3, 0 >$, place the vector $< 0, 0, 4 >$. What is the length of this vector?

c. Place the vector $< 2, 3, 0 > + < 0, 0, 4 > = < 2, 3, 4 >$ so that the three vectors $< 2, 3, 0 >$, $< 0, 0, 4 >$, and $< 2, 3, 4 >$ form a right triangle. Draw this right triangle on a sheet of paper, indicating which lengths are known.

d. Use the Pythagorean Theorem to obtain the length of the vector $< 2, 3, 4 >$.

Solution
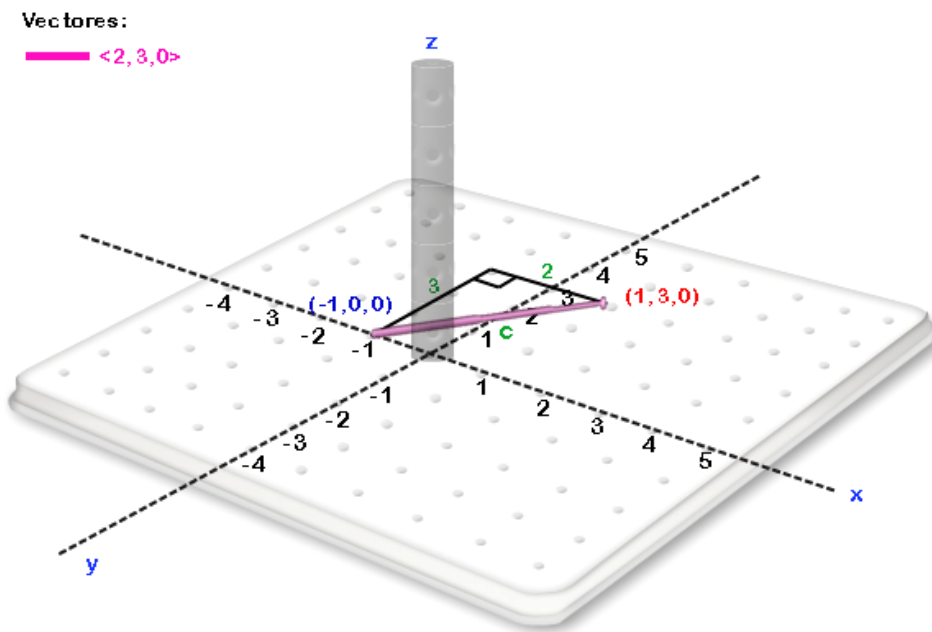
Vectores:
▬▬ <2,3,0>

a. $c = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$
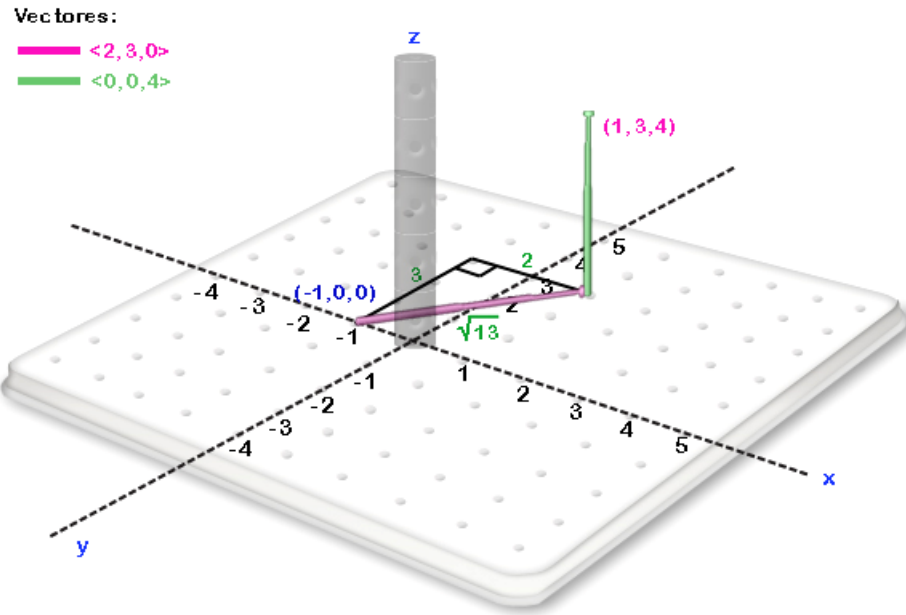
b.

c.

d. $d = \sqrt{\left(\sqrt{13}\right)^2 + 4^2} = \sqrt{13 + 16} = \sqrt{29}$

2.4 UNIT VECTORS AND NORMALIZATION

DEFINITION

A unit vector is a vector with length equal to 1.

Examples

Vector $\langle 0,1,0 \rangle$ has length $\sqrt{0^2+1^2+0^2} = 1$ and so it is a unit vector.

Vector $\langle 0.6,0,0.8 \rangle$ has length $\sqrt{(0.6)^2+0^2+(0.8)^2} = \sqrt{0.36+0.64} = 1$, so it is a unit vector.

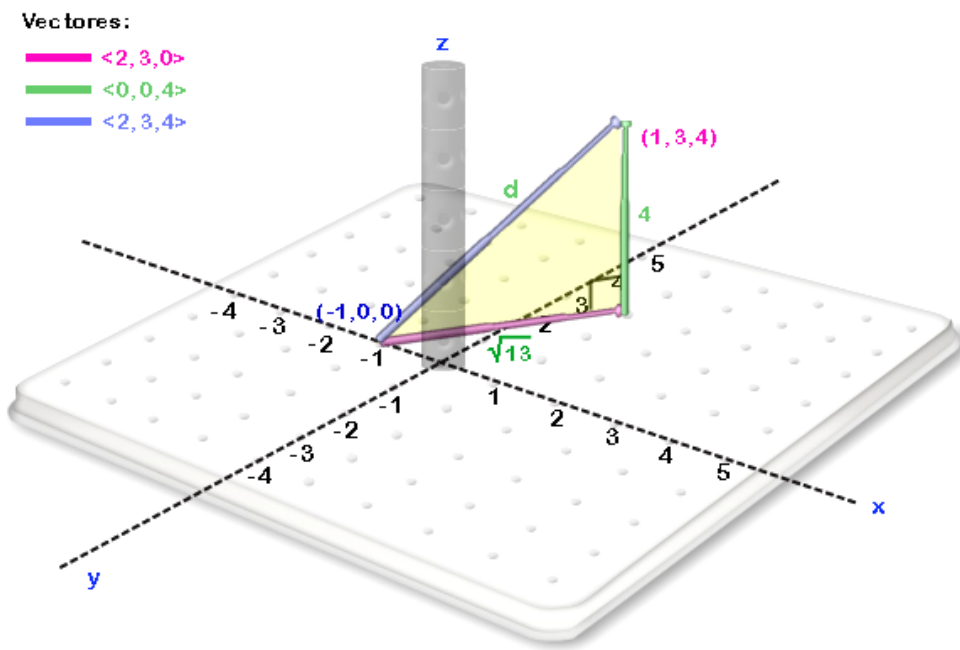Vector $\langle -1,3,2 \rangle$ has length $\sqrt{(-1)^2+3^2+2^2} = \sqrt{1+9+4} = \sqrt{14}$. Hence it is <u>not</u> a unit vector.

Example Exercise 2.3.1: Obtaining unit vectors

Find a unit vector that points in the direction of $\langle -3,0,4 \rangle$.

Vector $\langle -3,0,4 \rangle$ has length $\sqrt{(-3)^2+0^2+4^2} = \sqrt{25} = 5$. Therefore if we want a vector with length 1 we need to shrink $\langle -3,0,4 \rangle$ to one fifth of its size. Hence, the solution is:

$$(1/5)\langle -3,0,4 \rangle = \langle -3/5,0,4/5 \rangle .$$

To verify, we need only check the length of $\langle -3/5,0,4/5 \rangle$ as we know that multiplying by a positive scalar will shrink or extend a vector but will not change its direction. The length of $\langle -3/5,0,4/5 \rangle$ is $\sqrt{(-3/5)^2+0^2+(4/5)^2} = \sqrt{9/25+16/25} = \sqrt{25/25} = \sqrt{1} = 1$. Hence $\langle -3/5,0,4/5 \rangle$ is a unit vector that points in the direction of $\langle -3,0,4 \rangle$. See Figure 2.4.1.
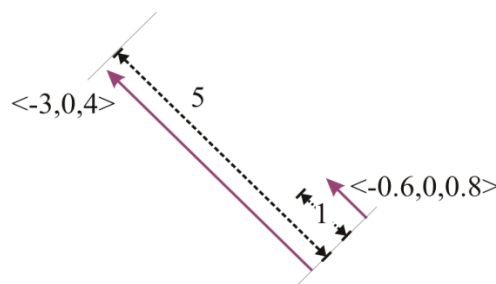

Figure 2.4.1

The procedure of finding a unit vector in a given direction is sometimes referred to as *normalizing* the vector. In general, if a vector $\vec{a}$ has length $|\vec{a}|$, multiplying the vector by the reciprocal of its length, $1/|\vec{a}|$, will not change the direction and will change the length of the vector to 1. The length of vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and so we have:

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Normalization of Vector $\langle a_1, a_2, a_3 \rangle$

A vector pointing in the direction of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ with a length of 1 is:

$$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \langle a_1, a_2, a_3 \rangle .$$

</div>

The unit vector $\frac{1}{|\vec{a}|}\vec{a}$ is commonly written as $\frac{\vec{a}}{|\vec{a}|}$.

Practice Problems

PP 2.4.1

    a. What is the length of vector $< 0, -4, 3 >$?

    b. What fraction of $< 0, -4, 3 >$ do we want to have a unit vector?

    c. Find a unit vector with direction equal to that of $< 0, -4, 3 >$.

Solution

    a. $\left| < 0, -4, 3 > \right| = \sqrt{(-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

    b. The fraction is $\frac{1}{5}$ of $<0, -4, 3>$

    c. The vector is $\frac{1}{5}<0, -4, 3> = < 0, \frac{-4}{5}, \frac{3}{5} >$.

Exercises:

1. Illustrate the following:

   a. Use your 3D Kit to place both vectors $\vec{a} = \langle 3,3,1 \rangle$ and $\vec{b} = \langle 2,5,3 \rangle$ starting at the point $P(1,1,3)$.

   b. Now starting at point $Q(3,4,6)$ represent a displacement of $_{-\vec{b}}$ followed by a displacement of $_{\vec{a}}$. Where do you end?

   c. Use part (b) to represent vector $_{-\vec{ba}}$ $^-$. Is this different than vector $_{\vec{ab}}$ $^-$?

2. For each of the following sets of vectors,

   i. Place $\vec{a}$ and $\vec{b}$ on your 3D kit originating from the point (1, 1, 2)

   ii. Find $|\vec{a}|$ and $|\vec{b}|$

   iii. Find unit vectors that point in the directions $\vec{a}$ and $\vec{b}$

   iv. Place the vectors $-\vec{a}, 3 \cdot \vec{b}, \vec{a}+\vec{b}, \vec{a}-\vec{b}$ and $\vec{b}-\vec{a}$ on your 3D kit

   a. $\vec{a} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$

   b. $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

   c. $\vec{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

   d. $\vec{a} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$

   e. $\vec{a} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

3. Given that $\vec{a} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$, express the following vectors in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

   a. The vector that goes from the point (-3, 1 , 2) to the point (3, 1, 2).
   b. The vector that goes from the point (2, -1 , 4) to the point (2, 9, 4).
   c. The vector that goes from the point (2, -1 , 3) to the point (1, 1, 1).
   d. The vector that goes from the point (-2, 1 , 6) to the point (4, 6, -2).
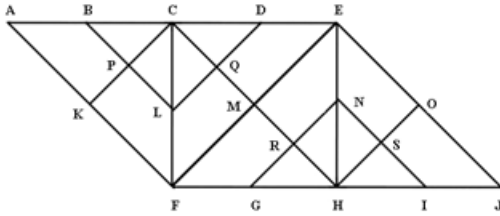
4. In the following diagram, $\vec{a} = \overrightarrow{AE}$ (I.e., the vector that goes from point A to point E) and $\vec{b} = \overrightarrow{AI}$
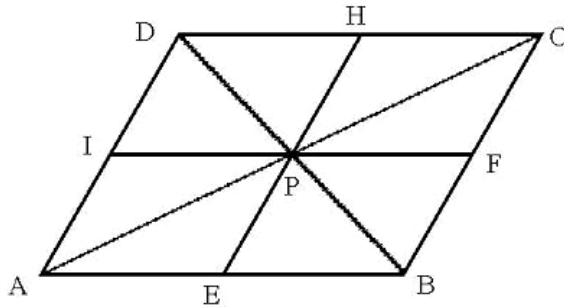
Express the following vectors in terms of $\vec{a}$ and $\vec{b}$

i. $\overrightarrow{EP}$ iv. $\overrightarrow{AF}$ vii. $\overrightarrow{HE}$

ii. $\overrightarrow{AP}$ v. $\overrightarrow{EC}$ viii. $\overrightarrow{CA}$

iii. $\overrightarrow{BP}$ vi. $\overrightarrow{ED}$ ix. $\overrightarrow{FD}$

5. In the following diagram, $\vec{a} = \overrightarrow{HN}$ and $\vec{b} = \overrightarrow{HR}$

Express the following vectors in terms of $\vec{a}$ and $\vec{b}$

i. $\overrightarrow{HF}$ iv. $\overrightarrow{EF}$ vii. $\overrightarrow{HJ}$

ii. $\overrightarrow{GN}$ v. $\overrightarrow{GC}$ viii. $\overrightarrow{CA}$

iii. $\overrightarrow{BL}$ vi. $\overrightarrow{ED}$ ix. $\overrightarrow{FD}$